Maclaurin's Expansion

1.

(a) Using standard series from the List of Formulae (MF26), show that the first three non-zero terms in the Maclaurin expansion of $\sec x$ are $1+\frac{x^2}{2}+\frac{5x^4}{24}$. [3]

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(a)

From MF26:

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots+\frac{(-1)^r x^{2r}}{(2r)!}+\ldots \qquad \text{(all } x\text{)}$$

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\ldots+\frac{n(n-1)\ldots(n-r+1)}{r!}x^r+\ldots \qquad (|x|<1)$$

$$\sec x = \frac{1}{\cos x}$$
$$= (\cos x)^{-1}$$
$$= \left(1-\frac{x^2}{2}+\frac{x^4}{24}+\cdots\right)^{-1}$$
$$= 1+(-1)\left(-\frac{x^2}{2}+\frac{x^4}{24}\right)+\frac{-1(-2)}{2}\left(-\frac{x^2}{2}+\frac{x^4}{24}\right)^2+\cdots$$
$$= 1+\frac{x^2}{2}-\frac{x^4}{24}+\frac{x^4}{4}+\cdots$$
$$= 1+\frac{x^2}{2}+\frac{5x^4}{24}+\cdots \text{ (Shown)}$$

2.

(b) A curve is defined by the equation

$$\left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}+xy=\sqrt{1+x^2}$$

and (0, 1) is a point on the curve.

(i) Find the Maclaurin's expansion of y up to and including the term in x^2. [3]

(ii) Hence, find the series expansion of e^y, up to and including the term in x^2. [3]

{(i) $1+x-\frac{x^2}{2}+\cdots$ (ii) $e(1+x)$}

(b)

(i) $$(1+x^2)\frac{dy}{dx}+xy=\sqrt{1+x^2}$$
$$(1+x^2)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+x\frac{dy}{dx}+y=\frac{1}{2}(1+x^2)^{-\frac{1}{2}}(2x)$$
$$(1+x^2)\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y=\frac{x}{\sqrt{1+x^2}}$$

$x=0$:

$f(0)=1$

$f'(0)=1$

$f''(0)=-1$

$$y=f(0)+xf'(0)+\frac{x^2}{2}f''(0)+\cdots$$
$$=1+x-\frac{x^2}{2}+\cdots \text{ (Ans)}$$

(ii)

$$e^y = e^{1+x-\frac{x^2}{2}+\cdots}$$
$$\approx e\times e^{x-\frac{x^2}{2}}$$
$$= e\left[1+x-\frac{x^2}{2}+\frac{\left(x-\frac{x^2}{2}\right)^2}{2}+\cdots\right]$$
$$= e\left(1+x-\frac{x^2}{2}+\frac{x^2}{2}+\cdots\right)$$
$$= e(1+x) \text{ (Ans)}$$

3.

22. JJC/2018/2/1

Given that $f(x)=e^{\sin x}$, use the standard series to find the series expansion for $f(x)$ in the form $a+bx+cx^2+dx^3$, where a, b, c and d are constants to be determined.

Hence show that the first three non-zero terms for the expansion of $\frac{1}{\left(e^{\sin x}\right)^2}$ in ascending powers of x is $1-2x+2x^2$. [4]

The function $y=g(x)$ satisfies $4\frac{dy}{dx}=(y+1)^2$ and $y=1$ at $x=0$.

(i) By further differentiation, find the series expansion for $g(x)$, up to and including the term in x^3. Hence show that when x is small, $g(x)-f(x)\approx\frac{1}{4}x^3$. [5]

(ii) By using the result in **(i)**, justify whether $f(x)$ is a good approximation to $g(x)$ for values of x close to zero. [1]

$\{1+x+\frac{x^2}{2}+0x^3+\cdots$; (i) — (ii) approximation is good$\}$

Q22.

$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots+\frac{(-1)^r x^{2r+1}}{(2r+1)!}+\ldots$ (all x)

$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^r}{r!}+\ldots$ (all x)

$$f(x)=e^{\sin x}$$
$$=e^{x-\frac{x^3}{6}+\cdots}$$
$$=1+x-\frac{x^3}{6}+\cdots+\frac{\left(x-\frac{x^3}{6}+\cdots\right)^2}{2}+\frac{\left(x-\frac{x^3}{6}+\cdots\right)^3}{6}+\cdots$$
$$=1+x-\frac{x^3}{6}+\frac{x^2-\frac{1}{3}x^4+\cdots}{2}+\frac{x^3+\cdots}{6}+\cdots$$
$$=1+x+\frac{x^2}{2}+0x^3+\cdots \text{ (Ans)}$$

$$\frac{1}{(e^{\sin x})^2}=\left(=1+x+\frac{x^2}{2}+0x^3+\cdots\right)^{-2}$$
$$=1+(-2)\left(x+\frac{x^2}{2}\right)+\frac{-2(-3)}{2}\left(x+\frac{x^2}{2}\right)^2+\cdots$$
$$=1-2x-x^2+3x^2+\cdots$$
$$=1-2x+2x^2+\cdots \text{ (Shown)}$$

(i)

$$y=g(x)$$
$$4\frac{dy}{dx}=(y+1)^2$$
$$4\frac{d^2y}{dx^2}=2(y+1)\frac{dy}{dx}$$
$$8\frac{d^2y}{dx^2}=4(y+1)\frac{dy}{dx}=(y+1)^3$$
$$8\frac{d^3y}{dx^3}=3(y+1)^2\frac{dy}{dx}=\frac{3}{4}\times 4(y+1)^2\frac{dy}{dx}$$
$$=\frac{3}{4}(y+1)^4$$

$x=0$:

$g(0)=1$

$g'(0)=1$

$g''(0)=1$

$g'''(0)=\frac{3}{2}$

$$g(x)=g(0)+xg'(0)+\frac{x^2}{2}g''(0)+\frac{x^3}{6}g'''(0)+\cdots$$
$$=1+x+\frac{x^2}{2}+\frac{1}{4}x^3+\cdots$$
$$g(x)-f(x)=1+x+\frac{x^2}{2}+\frac{1}{4}x^3+\cdots-\left(1+x+\frac{x^2}{2}+0x^3+\cdots\right)$$
$$\approx\frac{1}{4}x^3 \text{ (Shown)}$$

(ii)

$g(x)-f(x)\approx\frac{1}{4}x^3$

As $x\to 0$:

$\frac{1}{4}x^3\to 0$

$\Rightarrow$ $g(x)-f(x)\approx 0$

$\Rightarrow$ $g(x)\approx f(x)$

$\therefore$ The approximation is good (Justified)